arXiv:quant-ph/9705008v2 8 May 1997

Coupling Classical and Quantum Variables using Continuous Quantum Measurement Theory

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Preprint IC 96-97/46, quant-ph/9705008. May, 1997

Submitted to *Physical Review Letters*

ABSTRACT: We propose a system of equations to describe the interaction of a quasiclassical variable X with a set of quantum variables x that goes beyond the usual mean field approximation. The idea is to regard the quantum system as continuously and imprecisely measured by the classical system. The effective equations of motion for the classical system therefore consist of treating the quantum variable x as a stochastic c-number $\bar{x}(t)$ the probability distibution for which is given by the theory of continuous quantum measurements. The resulting theory is similar to the usual mean field equations (in which x is replaced by its quantum expectation value) but with two differences: a noise term, and more importantly, the state of the quantum subsystem evolves according to the stochastic non-linear Schrödinger equation of a continuously measured system. In the case in which the quantum system starts out in a superposition of well-separated localized states, the classical system goes into a statistical mixture of trajectories, one trajectory for each individual localized state.

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A variety of problems in a number of different fields involve coupling quantum variables to variables that are effectively classical. A case of particular interest is quantum field theory in curved space time, where one would often like to understand how a quantized matter field affects a classical gravitational field. The most commonly postulated way of modeling this situation is the semiclassical Einstein equations [1,2]:

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle. \tag{1}$$

Here, the left hand side is the Einstein tensor of the classical metric field $g_{\mu\nu}$ and the right hand side is the expectation value of the energy momentum tensor of a quantum field. Much effort has been put into finding solutions to Eq.(1), especially in the context of black hole physics.

Yet one cannot realistically expect that an equation such as (1) could be valid in more than a very limited set of circumstances. One would expect it to be valid, for example, only when the fluctuations in energy density are small [3,4], and it is not difficult to produce situations in which its predictions are not physically reasonable [5,6]. In particular, when the quantum state of the matter field consists of a superposition of two well-separated localized states, Eq.(1) suggests that the gravitational field couples to the average energy density of the two states, whilst physical intuition suggests that the gravitational field feels the energy of one or other of the localized matter states, with some probability. It therefore becomes of interest to ask, is there a way of going beyond the naive mean field equations which sensibly accommodates a wide class of non-trivial matter states, but without having to tackle the considerably more difficult question of quantizing the gravitational field?

In this letter we will present a simple scheme for coupling classical and quantum variables which goes far beyond the naive mean field equations, and produces intuitively sensible results in the key case of superposition states. We will not address the full problem of

the semiclassical Einstein equations (1), but rather, we will concentrate on a simple model in which the scheme is easily presented and perhaps verified. Our attempt to describe the coupling of classical and quantum variables is of course one of many [7,8,9].

We consider a classical particle with position X in a potential $V(X)$ coupled to a harmonic oscillator with position x which will later be quantized. The action is

$$S = \int dt \ \left(\frac{1}{2} M \dot{X}^2 + V(X) + \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 - \lambda X x \right). \tag{2}$$

Hence the classical equations of motion are

$$M \ddot{X} + V'(X) + \lambda x = 0, \tag{3}$$

$$m \ddot{x} + m \omega^2 x + \lambda X = 0. \tag{4}$$

The naive mean field approach involves replacing (3) with the equation

$$M \ddot{X} + V'(X) + \lambda \langle \psi | \hat{x} | \psi \rangle = 0, \tag{5}$$

and replacing (4) with the Schrödinger equation

$$\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} \left(\hat{H}_0 + \lambda X \hat{x} \right) |\psi\rangle \tag{6}$$

for the quantum particle. $\hat{H}_0$ is the Hamiltonian of the quantum particle (in this case a harmonic oscillator) and $-X(t)$ is regarded as an external classical force. As stated above, however, we do not expect this scheme to be valid very widely.

Our new approach is to think of the classical particle as in some sense "measuring" the quantum particle's position and responding to the measured c-number result $\bar{x}$. (A precursor to this idea may be found in Ref.[10]).

Consider first, therefore, the consequences of standard quantum measurement theory for the evolution of the coupled classical and quantum systems over a small interval of

time δt. The state $|\psi\rangle$ of the quantum system will evolve, as a result of the measurement, into the (unnormalized) state

$$|\Psi_{\bar{x}}\rangle = \hat{P}_{\bar{x}} e^{-i\hat{H}\delta t}|\psi\rangle \tag{7}$$

where $\hat{H} = \hat{H}_0 + \lambda X \hat{x}$ and $\hat{P}_{\bar{x}}$ is a projection operator which asks whether the position of the quantum particle is $\bar{x}$, to within some precision. The probability that the measurement yields the result $\bar{x}$ is given by $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$. It is then natural to suppose that the classical particle, in responding to the measured result, will evolve during this small time interval according to the equation of motion

$$M\ddot{X} + V'(X) + \lambda\bar{x} = 0, \tag{8}$$

with probability $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$.

Now we would like to repeat the process for an arbitrary number of time steps and then take the continuum limit. If $\hat{P}_{\bar{x}}$ is an exact projection operator, *i.e.*, one for which $\hat{P}_{\bar{x}}^2 = \hat{P}_{\bar{x}}$, the continuum limit is trivial and of no interest (this is the watchdog effect). However, standard quantum measurement theory has been generalized to a well-defined and non-trivial process that acts continuously in time by replacing $\hat{P}_{\bar{x}}$ with a positive operator-valued measure (POVM) [11,12,13,14,10]. The simplest example, which we use here, is a Gaussian,

$$\hat{P}_{\bar{x}} = \frac{1}{(4\pi\Delta^2)^{\frac{1}{2}}} \ \exp\left(-\frac{(\hat{x}-\bar{x})^2}{4\Delta^2}\right) \tag{9}$$

and the continuum limit involves taking $\Delta\to\infty$ as $\delta t\to 0$ in such a way that $\Delta^2\delta t$ is held constant. The evolution of the wave function of the quantum system is then conveniently expressed in terms of a path-integral expression for the unnormalized wave function:

$$\Psi_{[\bar{x}(t)]}(x',t') = \int \mathcal{D}x \exp\left(\frac{i}{\hbar}\int_0^{t'} dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \lambda x X\right)\right)$$
$$\times \exp\left(-\frac{\lambda^2}{\hbar^2}\int_0^{t'} dt \frac{(x-\bar{x})^2}{4\sigma^2}\right) \ \Psi(x_0,0). \tag{10}$$

Here, the integral is over paths $x(t)$ satisfying $x(0) = x_0$ and $x(t') = x'$. The classical particle at each moment of time evolves according to Eq.(8), where the functional probability distribution of the entire measured path $\bar{x}(t)$ takes the form:

$$p[\bar{x}(t)] = \langle \Psi_{[\bar{x}(t)]} | \Psi_{[\bar{x}(t)]} \rangle. \tag{11}$$

The formula (10) contains an undetermined parameter σ tuning the precision of the measurement. (We have chosen to include the factor of λ^2 in the parametrization of the width of the measurement in order to conveniently accommodate the physically expected feature that (10) must reduce to unitary evolution when $\lambda = 0$). To obtain a plausible estimate as to the value of σ, consider what it means to be classical. We take the point of view that there are no *fundamentally* classical systems in the world, only quantum systems that are effectively classical under certain conditions. The most important condition that needs to be satisfied for a system to be essentially classical is *decoherence* – interference between histories of certain types of variables (in this case position) must be destroyed (see, for example, Refs.[15,16,17]). Decoherence is typically brought about by some kind of coarse-graining procedure. A commonly used procedure is to couple to a heat bath and then trace it out. Whichever method is used to produce decoherence, a generic result is that the decohered variables are subject to fluctuations (as a result of the interaction with the heat bath, for example). In contrast to fundamentally classical systems, effectively classical systems therefore always suffer a minimal amount of imprecision due to these fluctuations. (There are also of course the ubiquitous quantum fluctuations but these are typically much smaller). The consequence of this is that the effectively classical system will necessarily be limited in the precision with which it can measure the quantum system, because of its own intrinsic imprecision. An estimation of σ ought therefore to be possible from the fluctuations in the classical system.

To be concrete, suppose that in the absence of a coupling to the quantum particle, the classical particle suffers an imprecision ΔF in the degree to which the classical field equations are satisfied. This means that the distribution of $X(t)$ is expected to be proportional to the Gaussian functional [15,18],

$$\exp\left(-\frac{1}{2(\Delta F)^2}\int dt\left(M\ddot{X}+V'(X)\right)^2\right). \tag{12}$$

An example is the case of a thermal environment producing the decoherence and fluctuations in X, in which case $(\Delta F)^2$ is of order $M\gamma k_B T$, where γ is the dissipation of the environment and T its temperature. On dimensional grounds, given the coupling between X and x in (2), a reasonable choice for σ is $\sigma \sim \Delta F/\hbar$, (so $\sigma^2 \sim M\gamma k_B T/\hbar^2$ in the case of thermal fluctuations). We will see further evidence for this choice below.

The scheme is therefore as follows. We solve the equations (8) and (10) where $\bar{x}(t)$ is regarded as a stochastic variable whose probability distribution is given by (11). The final result is therefore an ensemble of $\bar{x}$-dependent classical and quantum trajectories respectively for the two particles, with an interdependent probability distribution.

It turns out that this system (8), (10), (11) can be rewritten in such a way that brings it closer to the form of the naive mean field equations (5), (6). The basic issue is that Eq.(11) gives the probability for an entire history of measured alternatives, $\bar{x}(t)$. Yet the naive mean field equations (5), (6) are evolution equations defined at each moment of time. This therefore leads one to ask, is it possible to rewrite the system (8), (10), (11) in terms of evolution equations?

This is indeed possible. Consider the basic process (7) with the Gaussian projector (9), but in addition let the the state vector be normalized at each time step. Then denoting the normalized state at each time by $|\psi\rangle$, and taking the continuum limit in the manner indicate above, it is readily shown [13] that $|\psi\rangle$ obeys a stochastic non-linear equation

describing a system undergoing continuous measurement:

$$\frac{d}{dt}|\psi\rangle = \left(-\frac{i}{\hbar}(\hat{H}_0 + \lambda X\hat{x}) - \frac{\lambda^2}{4\hbar^2\sigma^2}(\hat{x} - \langle\hat{x}\rangle)^2\right)|\psi\rangle + \frac{\lambda}{2\hbar\sigma}(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle\eta(t). \tag{13}$$

Here, $\eta(t)$ is the standard Gaussian white noise, with linear and quadratic means,

$$\langle\eta(t)\rangle_S = 0, \qquad \langle\eta(t)\eta(t')\rangle_S = \delta(t - t'). \tag{14}$$

where $\langle\ \rangle_S$ denotes stochastic averaging. The noise terms are to be interpreted in the sense of Ito. The measured value $\bar{x}$ is then related to η by

$$\bar{x} = \langle\psi|x|\psi\rangle + \frac{\hbar\sigma}{\lambda}\eta(t). \tag{15}$$

Hence, the final equations that replace (5), are

$$M\ddot{X} + V'(X) + \lambda\langle\psi|\hat{x}|\psi\rangle + \hbar\sigma\eta(t) = 0 \tag{16}$$

and (6) is replaced by the stochastic non-linear equation (13). Referring back to our explanation of classicality, we see that, indeed, the classical particle suffers a random force independent of its coupling to the quantum particle, leading formally to the distribution (12). In the case of a thermal environment, the random force should be $\sqrt{2M\gamma k_B T}\eta(t)$, in order to coincide with the standard Langevin equation of classical Brownian motion, and this is indeed the case (if the numerical factor in the choice of σ is chosen so that $\sigma^2 = 2M\gamma k_B T/\hbar^2$).

There are two differences between the system, (13)–(16) and the naive mean field equations. One is the noise term, η. In Eq.(16) the noise clearly describes fluctuations about the naive semiclassical trajectories. This sort of modification to the semiclassical Einstein equations has been considered previously [4,19].

More important is the novelty that the state $|\psi\rangle$ evolves according to the stochastic non-linear equation (13), and hence its evolution is very different to that under the usual

Schrödinger equation. In particular, it may be shown that all solutions to (13) undergo *localization* [20,21,22,23,24,25] on a time scale which might be extremely short compared to the oscillator's freqency ω. That is, every initial state rapidly evolves to a generalized coherent state centred around values $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$ undergoing classical Brownian motion. (The results cited above are readily extended to the case here in which the Hamiltonian contains a linear coupling to an external force $-X(t)$). Which particular solution the state becomes centred around depends statistically on the initial state of the system. For an initial state consisting of a superposition of well-separated coherent states,

$$|\psi\rangle = \alpha_1 |x_1 p_1\rangle + \alpha_2 |x_2 p_2\rangle \tag{17}$$

the state after localization time will, with probability $|\alpha_1|^2$, be as if the initial state were just $|x_1 p_1\rangle$, and with probability $|\alpha_2|^2$, will be as if the initial state were just $|x_2 p_2\rangle$ [24,25]. The localization time $\sim 1/\sigma^2 (x_1 - x_2)^2$ becomes, with our previous choice $\sigma^2 \sim M \gamma k_B T / \hbar^2$, very short indeed if the classical particle has a large mass M.

Hence in the new semiclassical equations (13)–(16), effectively what happens is that we solve separately for the two initial states $|x_1 p_1\rangle$ and $|x_2 p_2\rangle$, and the classical particle then follows the first solution with probability $|\alpha_1|^2$ and the second with probability $|\alpha_2|^2$.

In simple terms, therefore, an almost classical system interacting through position with a quantum system in a superposition state (17), "sees" one or other of the superposition states, with some probability, and not the mean position of the entire state. This is the key case for which the naive mean field equations fail to give intuitively sensible results [6,26].

It is interesting to note that non-linear Schrödinger equations have been considered before in the context of the semiclassical Einstein equations [5,27,28], because the combined system consiting of (1) together with the Schrödinger equation for the quantum state is

non-linear. The motivation here is rather different. The equation (13) used here arises because it gives a phenomenological description of continuous measurement.

Similar results are obtained with different types of couplings, for example to momentum or to energy [29]. Obviously an important challenge is to extend to quantum field theories and hence to obtain a generalization of Eq.(1). This would mean confronting the difficult issues of covariance and non-renormalizability. This will be discussed elsewhere.

We have presented a scheme for coupling classical and quantum variables which appears to be reasonable on physical grounds and give intuitively sensible results. It is based on the premise that the interaction between the classical and quantum variables may be regarded as a quantum measurement. The mathematics of continuous quantum measurement theory then fixes the overall structure of the scheme, but an additional physical argument is required to fix the parameter describing the precision of the measurement.

In the simple model we considered here, the quantum theory of the whole closed system (including the possible environmental degrees of freedom) exists. In this case it is therefore reasonable to explore the possibility that the scheme presented here emerges as an effective theory under suitable conditions (taking into account the requirements for the classicality above). In a longer more detailed paper it will be shown, using the decoherent histories approach to quantum theory, how an effective theory very similar to this scheme may arise [29]. This confirms the physical arguments given above for choice of the width σ of the continuous measurements. Furthermore, the theory of continuous quantum measurements is in fact closely related to the so-called hybrid representation of composite quantum systems [8,30,31], thus providing an alternative framework for examining the emergence of the scheme.

We have presented the scheme here without appealing to more fundamental origins

because we believe that it stands on its own terms as a simple and plausible phenomenological model. Furthermore, since the underlying quantum theory of the classical system may not be known, it is important to understand how such phenomenological models are constructed directly. We do not claim, however, that our scheme eliminates all known controversies of the naive mean field method or of non-linear quantum theories generally [32].

ACKNOWLEDGEMENTS

L.D. was supported by a Visiting Research Fellowship from EPSRC and by grant number OTKA T016047. We are grateful to Chris Isham for useful comments.

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